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ELEMENTS OF CONSTRUCTIVE GEOMETRY

WILLIAM NOETLING

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ELEMENTS

OF

CONSTRUCTIVE GEOMETRY

INDUCTIVELY PRESENTED

BY

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AUTHOR OF "NOTES ON THE SCIENCE AND ART OF EDUCATION"

FROM THE GERMAN OF K. H. STÖCKER

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PREFACE.

BELIEVING that there is room in our schools for an elementary inductive work on geometry, I have taken the matter of one of the best German books and, as much as possible, changed the method from the deductive to the inductive.

The method is constructive, the pupils drawing their own figures. It begins with simple exercises, which increase in difficulty by almost imperceptible steps.

The treatment of the subject will, I think, be found thoroughly educative and practical, and an excellent introduction to demonstrative geometry.

The introduction of this kind of geometry into the country schools, where geometry has heretofore not been taught, and into the lower grades of village and city schools, will furnish a sort of knowledge to the pupils of those schools that will prove valuable to them in the various callings of life; a knowledge, too, of which they are now left in total ignorance.

The work is intended for children from ten, or even nine, years of age upward.

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It may seem to some into whose hands this book falls that diagrams should have been given here and there in the text; but I believe that intelligent teachers would rather not have them in the book, and therefore I have omitted them. The figures whose construction may puzzle some teachers a little, can be found in almost all of the large dictionaries.

The instruments that are needed to make the constructions and measurements are a foot rule, a pair of compasses, and a semi-circular protractor.

WILLIAM NOETLING.

STATE NORMAL SCHOOL,
BLOOMSBURG, PA., June 1, 1897.

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ELEMENTS OF

CONSTRUCTIVE GEOMETRY.

I. LINES.

1. Draw a free-hand line (*i.e.* without ruler) upon the blackboard or paper, making it, according to your judgment, one inch in length. Compare its length with an inch measure.

NOTE. — All lines not otherwise designated are to be considered straight.

2. Draw free-hand lines two, three, four, etc., inches in length, and compare their lengths with the corresponding measures.

3. Draw a free-hand line of given length, and compare it with its corresponding measure; then bisect it and trisect it, and compare the lengths of the divisions.

4. Draw lines of various lengths, divide them into two, three, four, five, six, etc., equal parts, and compare the parts of each.

5. Suspend a small weight from the end of a cord, and draw a line on the blackboard that shall take the same direction as the cord. What name is applied to lines that take this direction?

6. What name is applied to a line that takes the direction of the surface of standing water? What is the difference between *horizontal* and *level?*

7. Draw a vertical line and a horizontal line upon the blackboard. Can you draw a line that is neither horizontal nor vertical? Draw such a line. What name is applied to it?

8. If I should mark a point upon the blackboard, and ask you to draw a line through it, would you know what direction it should take? If I should mark two points, and tell you to draw a line through them, would you know in what direction to draw it? How many points are necessary to determine the direction of a straight line? Would more points be of any service? In either case, why?

9. What name would you give to a line made partly of straight and partly of curved parts? What, to one made of straight parts taking different directions?

10. If you were required to go to a certain place, and had the choice of three roads — one of them partly curved and partly straight, another of straight parts taking different directions, and the third entirely straight — which one of them could you travel in the shortest time? Why? What kind of line is the shortest distance between two points?

11. How could you determine the length of a *mixed line?* How of a *broken line?*

REMARK. — Lines are designated by means of letters placed at their extremities. Thus, a line with the letter a at one extremity and b at the other, is termed the line ab.

12. Draw two lines that shall throughout their lengths be equally distant from each other. What name is applied to such lines? May more than two lines be *parallel* to one another?

REMARK. — *Diverging lines* are such as separate more and more the farther they are produced, and *converging lines* such as approach each other more and more the farther they are extended.

II. ANGLES.

1. Draw two converging lines, and extend them until they meet. The opening between the lines is called an *angle*, the lines are the *sides* of the angle, and the point in which the lines meet is its *vertex*.

2. Angles are named by means of letters placed at their sides and at the vertex.

3. Place the letter a at the extremity of one of the sides of the angle you have formed, b (outside of the angle) at the vertex, and c at the extremity of the other side. Thus, the angle is named the angle abc; the middle letter being the one at the vertex.

4. An angle may also be named by a letter placed between its sides near their intersection.

5. With the vertex of the angle you have formed, as a center, describe a circle whose circumference shall cut both sides of the angle. The arc (part of circumference) of the circle between the sides is the *measure of the angle.*

6. With any radius, describe a circle. Divide its circumference into four equal parts, and from two of the points, a fourth of the circumference apart, draw lines to the center, forming a *center angle* whose measure is a fourth of the circumference. An angle whose measure is one fourth of a circumference is a *right angle*, and its sides are *perpendicular* to each other.

7. Draw a line, and at its middle point erect a perpendicular to it. How do the angles on each side of the perpendicular compare with each other in size? What kind of angles are they?

8. What does *acute* mean? What, *obtuse ?*

9. A center angle that is less than a right angle is an *acute* angle, and one that is greater is an *obtuse* angle.

10. Make an acute angle; also an obtuse angle. How do the sides of these angles meet each other?

11. All angles not right angles are *oblique angles.*

12. Make a center angle whose sides are a half circumference apart. What kind of line do the sides form? Such an angle is called a *straight angle.*

13. An angle whose measure is more than a half circumference is a *convex angle.* A convex angle presents a *corner* at the vertex; while right, acute, and obtuse angles present *angles.*

14. Every circle or circumference is supposed to be divided into 360 equal parts, called *degrees.* How many degrees in a right angle? In a straight angle?

15. Thirty degrees (written 30°) equal what part of a right angle?

16. An arc of 45° measures what kind of angle? What part of a right angle?

17. Name the angle measured by 75°; also that contained between two lines 120° apart.

18. One fifth of the circumference measures what kind of angle? One third measures what kind?

19. What kind of angle do the hour and minute hands of a clock make with each other at 3 o'clock? At 6? At 9?

20. How long does it take the minute hand to pass over the arc of an acute angle? Of a right angle? Of an obtuse angle? Of a straight angle? Of a convex angle?

21. If a man who had been going due east should change his direction three right angles to the left, in what direction

would he then be going? If he had been going north, and had changed his direction one right angle to the right, in what direction would he be going?

22. How many degrees in an angle that is 10°, 18°, 25°, or 60° less than a right angle?

23. How many degrees in an angle that is 15°, 24°, 35°, or 75° larger than a right angle?

24. A straight angle is how many times as large as an angle of 12°, 18°, 45°, or 60°?

25. How does a right angle compare with an angle of 270°? How does a straight angle compare with an angle of 270°? How does an angle 45° larger or smaller than a right angle compare with one of 270°? One $22\frac{1}{2}$° larger or smaller than a straight angle?

26. Draw an angle; also the arc that measures it. With the opening of the dividers, with which the arc was drawn, describe part of a circle; upon it lay off the measuring arc of the angle, and from its center draw lines to its extremities, thus forming a second angle. How do the two angles compare in size?

NOTE. — All angles of the same problem or figure (geometrical) should have their measures described with the same opening of the dividers.

27. Make an angle of $22\frac{1}{2}$°. What part of a right angle is it?

28. Make two angles, of which the second shall be twice as large as the first.

29. Make an angle that is as large as the sum of a half and a fourth of a right angle.

30. Make two unequal angles, and a third that shall be equal to their difference.

31. Make an obtuse angle. The angle between its sides is a *concave* or *hollow angle* (an angle less than a straight angle). What is the sum of the two angles, the concave and the convex? How many right angles in their sum? If the concave angle is 38°, 62°, 84°, or 125°, what is the convex angle? If the convex angle is 195°, 225°, or 312°, what is the concave angle?

32. Form several angles around a point, and find how many right angles their sum contains. How many right angles can be constructed around a point? How many degrees in all the angles around a point?

33. What does *adjacent* mean?

34. Construct an angle, and prolong one of its sides beyond the vertex, thus forming another angle adjacent to the first. What is the measure, or sum, of both?

NOTE. — Two angles that have a common side and the other two sides forming a straight line are *adjacent angles.*

35. What is the sum of any two adjacent angles?

36. When one of two adjacent angles is 21°, 75°, or 68°, what is the other?

37. What is the measure of each of two equal adjacent angles? How does the common side meet the others?

38. What is the difference between *perpendicular* and *vertical?* Illustrate it upon the blackboard.

REMARK. — Two angles, the sum of whose measures is one fourth of a circumference (90°), are *complements* of each other; and two, the sum of whose measures is a half circumference (180°), are *supplements* of each other.

39. Construct an angle, and prolong its sides beyond the vertex. The angle formed by the prolongation of the lines and the first angle are called *opposite angles.* How do the

opposite angles compare in size? Compare other opposite angles. What general inference may be drawn concerning opposite angles?

REMARK. — If the pupils have a sufficient knowledge of algebra, the relation of opposite and vertical angles may be shown by means of the equation.

40. When an angle adjacent to two opposite angles is 92°, 120°, 112°, or 146°, what is each of the opposite angles?

III. INTERSECTED PARALLELS.

1. When are two lines parallel to each other? Draw two parallel lines. Draw an oblique line across them. The oblique line is called a *transversal.*

2. How many angles does the transversal form with the parallels? Indicate each of the angles by a letter.

3. Name two angles that lie within the parallels and on the same side of the transversal. Angles that lie within the parallels are called *interior angles.* Name the other pair of interior angles.

4. Name two angles that lie on the same side of the transversal, but without the parallels. Angles that lie without the parallels are called *exterior angles.* Point out another pair of exterior angles.

5. Find two angles on the same side of the transversal — not at the same intersection — one within the parallels, the other without. Such angles are called *corresponding angles.* Name three other pairs of corresponding angles.

6. Which two interior angles lie on opposite sides of the transversal and at different intersections? Such angles are called *alternate interior angles.* Point out another pair of alternate interior angles.

7. What two exterior angles can you find that are neither on the same side of the transversal nor at the same intersection? Such angles are called *alternate exterior angles.* Find another pair of alternate exterior angles.

8. Find two angles, one exterior, the other interior, on different sides of the transversal and at different intersections. Such angles are called *alternate opposite* or *conjugate angles.*

9. Find two angles on the same side of the transversal, either both within or without the parallels, and at different intersections. Such angles are called *opposite angles.*

10. How do the measures of the corresponding angles compare? How, those of the alternate exterior angles? Of the alternate interior angles? What inference may be drawn concerning the corresponding, the alternate exterior, and the alternate interior angles of two parallels crossed by a transversal?

11. Describe a semicircle (half circle), and from one end of its curve, with the same opening of the dividers, lay off an angle equal to the sum of a pair of interior angles, and observe what part of the semicircumference it measures. In the same manner determine the measure of the sum of a pair of exterior, also of a pair of conjugate, angles. What do all these sums equal? What inference can you draw as to the sum of a pair of exterior, interior, or conjugate angles of two parallels crossed by a transversal?

12. If one of two adjacent angles is 94°, 116°, 108°, or 75°, how many degrees is the other?

13. If the adjacent angle of an alternate interior angle is 70°, 104°, 60°, or 45°, how many degrees are the alternate interior angles?

14. If the adjacent angle of an alternate exterior angle is 60°, 94°, 128°, or 165°, how many degrees are the alternate exterior angles?

15. The adjacent angle of an interior angle is 52°, 35°, 64°, or 28°; how many degrees in each of the interior angles?

16. The adjacent angle of an exterior angle is 88°, 120°, 109°, or 155°; how many degrees in each of the exterior angles?

17. The adjacent angle of a conjugate angle is 15°, 32°, 50°, or 82°; what is the measure of each of the conjugate angles?

18. If one of a pair of conjugate angles is 58°, 95°, 112°, or 150°, what is the other? Why?

19. If one of a pair of opposite angles is 80°, 120°, 46°, or 160°, what is the other? How can you tell?

20. Draw a transversal across two lines that are not parallel, and indicate the angles formed by letters or figures. Compare the measures of the corresponding angles on the diverging side; also those on the converging side. Compare the alternate interior angles; also the alternate exterior angles.

21. With the opening of the dividers (radius) used in comparing the foregoing angles, describe a circle. Lay off upon its circumference the sum of the interior angles on the diverging side of the transversal, and compare it with two right angles. In the same manner compare with two right angles the sum of the interior angles on the converging side, that of the exterior angles on the diverging side, the exterior angles on the converging side, and the conjugate angles.

22. By what relation of angles can you determine whether two lines crossed by a transversal are parallel? How could you determine if they were not crossed by a transversal?

23. Could you, from a point without a line, by means of angles, draw a parallel to the line? Explain how. Would a transversal help you to do it? How?

IV. TRIANGLES.

1. The Angles of Triangles.

1. Draw a triangle, and on the circumference of a circle determine the sum of its angles.

2. If one of the angles of a triangle were enlarged, what effect would the change have upon the other angles?

3. If two of its angles were enlarged, how would it affect the third? In either case, would the sum of the angles be changed? What general conclusion may therefore be drawn concerning the sum of the three angles of a triangle?

4. If two of the angles of a triangle are 65° and 52°, 27° and 74°, or 85° and 89°, what is the third angle?

5. How many angles of 90° (right angles) can a triangle contain? Draw such a triangle, and designate it *abc*, by writing one of these letters at the vertex of each angle. Which sides include the right angle? How do these sides meet each other?

6. Name the side opposite the right angle. This side is the *hypotenuse.*

7. Which is the longest side in every right-angled triangle?

8. What is the sum of the angles at the hypotenuse? What kind of angle is each of them?

9. How many degrees in one of the angles at the hypotenuse, when the other is 29°, 40°, 67°, or 71°?

10. How many obtuse angles can a triangle have?

11. Draw a triangle containing an obtuse angle.

12. In such a triangle, which of the sides, according to length, lies opposite the obtuse angle?

13. How does the sum of the angles at the longest side compare with 90°? What kind of angle, therefore, is each of them?

14. If the obtuse angle and one of the acute angles are 115° and 22°, 104° and 35°, or 129° and 31°, what is the third angle?

15. From a point without a line, draw a perpendicular to the line. From the same point, and on the same side of the perpendicular, draw also several oblique lines to the line.

16. What kind of a triangle do the perpendicular and nearest oblique line form?

17. How does the oblique line compare in length with the perpendicular? According to what inference?

18. What kind of triangle is that formed by the oblique lines?

19. How do the oblique lines compare with each other in length? According to what inference?

20. How many perpendiculars can be drawn from a point without a line to the line?

21. How many equal oblique lines can be drawn from a point without a line to the line? How many, from the same point, on one side of a perpendicular?

22. From a point without a line, draw two equal oblique lines to the line, and compare their distances from the foot of the perpendicular to the line. Compare also the distances

of two unequal oblique lines from the foot of the perpendicular.

23. What general inference can you draw concerning the distances of equal and of unequal lines from the foot of the perpendicular?

24. How many angles less than 90° (acute angles) can a triangle have? Draw such a triangle.

25. Two angles of an acute-angled triangle are 49° and 51°, 53° and 69°, or 80° and 79°; what is the third angle?

26. How many angles of a triangle may contain more than 180°, or be convex?

27. Draw a triangle. What is the sum of the interior and exterior angles at each corner? What, of all the corners? How much of this sum is derived from the interior angles? How much, from the exterior?

28. What inference can be drawn concerning the sum of the exterior angles of a triangle?

29. Draw a triangle, and prolong one of its sides beyond the vertex. The angle thus formed without the triangle is called an *exterior* (*outer*) *angle.* Find the difference between the exterior angle and the sum of the opposite interior angles.

30. What inference can you draw concerning the exterior angle of a triangle when compared with the sum of its two opposite interior angles?

31. The two opposite interior angles of a triangle are 70° and 65°, 43° and 82°, or 46° and 54°; what is the exterior angle?

32. The exterior angle is 134° and one of the opposite interior angles 52°, 84°, 56°, or 41°; what is the other?

2.—The Sides of Triangles.

a.—Triangles of Unequal Sides (Scalene Triangles).

1. Construct a triangle of three lines of which two together shall be shorter than the third. Do the same with three lines of which two together shall be equal to the third; also with three lines of which two together shall be greater than the third. What inference can you draw concerning the relation of the sum of any two sides of a triangle, when compared with the third side?

2. Draw a triangle of which no two sides shall be equal, and compare the angles. Are any of them equal? Which one (according to size) is found opposite the longest side? Which, opposite the next longest? Which, opposite the shortest? What inference can you therefore draw concerning the relation of the angles and their opposite sides, of a triangle?

b.—Triangles of Two Equal Sides (Isosceles Triangles).

1. Draw a triangle having two equal sides. The point in which the equal sides meet is the *vertex* of the triangle, and the side opposite the vertex is the *base.*

2. At which side do you find the angles that lie opposite the equal sides? Compare the measures of these angles. What is the inference concerning the angles at the base of an isosceles triangle?

3. An angle at the base is 48°, 63°, 27°, or 77°; what is the measure of the angle at the vertex?

4. The angle at the vertex is 96°, 78°, 44°, or 109°; what is each of the angles at the base?

5. From the vertex draw a line to the middle of the base, and compare the parts into which it divides the angle at

the vertex. Compare also the adjacent angles which it forms with the base. What kind of angles are they and how does the line meet the base? Into what kind of triangles does it divide the isosceles triangle?

6. *Can you erect a perpendicular to the line* ab *at the point* c, *using no instruments but a pair of compasses and a straight-edge or ruler?* If you should mark a point at equal distances on each side of *c*, and upon the line connecting the two points, as a base, construct an isosceles triangle, could you with its aid erect the perpendicular? Would a line from *c* to its vertex be the perpendicular? According to what inference?

7. *From a point* c, *without a line* ab, *draw a perpendicular to the line*, using the same instruments as in previous case. Would the isosceles triangle assist you in doing it? If so, explain how.

8. With the same instruments as before and the isosceles triangle, *divide a line into two equal parts.* Can you do it without the triangle? If so, how?

9. By means of the isosceles triangle and the instruments used before, *divide an angle into two equal parts.* If you can do it without the triangle, do so.

c. — Triangles of Three Equal Sides (Equilateral Triangles).

1. Construct a triangle of three equal sides. Measure the angles. How do they compare in size? Each one is what part of a right angle? How many degrees in each?

2. *At the end of a line, to erect a perpendicular to the line.* — Let *ab* be the line and *a* the point at which the perpendicular is to be erected. From *a* towards *b* lay off an assumed distance to *c*, as the base of an equilateral triangle, and construct the triangle. Mark the vertex of the tri-

angle *d*. Prolong *cd* to *e*, making *de* equal to *cd*, and draw a line from *e* to *a*.

3. What kind of triangle is *ade*? How many degrees in the angle *ead*? In *ade*? In *aed*? In *eac*? How do the lines *ae* and *ab* meet each other?

d. — The Congruence, or Agreement, of Triangles.

1. Construct two unequal-sided (scalene) triangles, making the sides of one of the same length as the corresponding sides of the other. Compare the angles opposite the longest sides of each, those opposite the next longest sides, and those opposite the shortest sides. What have you found about the angles lying opposite equal sides? If you should cut one of the triangles out, and lay it upon the other, in what respects would they agree? If two triangles agree in the lengths of their sides, what may be said of the triangles? (They are *congruent — agree in all like respects.*)

2. Construct two equal-sided angles, and connect the ends of the sides of each. You have thus formed two triangles of which two sides and the included angle of one are equal to two sides and the included angle of the other. Compare the remaining sides and angles. What may therefore be said of two triangles of which two sides and the included angle of one are equal to two sides and the included angle of the other?

3. Construct two triangles of which two angles and the included side of one shall be equal to two angles and the included side of the other, and compare the remaining like parts. What inference may be drawn concerning two such triangles?

4. Construct a triangle of which two sides shall be of unequal length. Construct an angle equal to that opposite

the longer of the two sides of the triangle, and make one of its sides equal to the shorter of the two sides. With the end of the shorter side as a center, and the longer side as a radius, describe a curve (arc of circle) which shall intersect the third side. The triangle thus constructed is equal to the other in two of its sides and the angle opposite the greater side. Compare the measures of the remaining similar parts. What inference may be drawn concerning two such triangles?

V. FOUR-SIDED FIGURES, OR QUADRILATERALS.

1. Construct a quadrilateral (four-sided figure). Lay off its angles successively on the arc of a circle, and find the number of degrees in their sum. What general inference may be drawn of the sum of the angles of a quadrilateral?

2. If three angles of a quadrilateral are 102°–78°–96°, 59°–120°–92°, or 67°–117°–101°, what is the fourth angle?

3. How many angles of 90° can a quadrilateral have? Construct such a quadrilateral.

4. How many angles greater than 90° (obtuse angles) can a quadrilateral have? Construct such a figure.

5. How many acute angles can a quadrilateral have?

6. If three angles of a quadrilateral are 89°–79°–69°, what kind of angle is the fourth? If three of the angles are 55°–45°–35°, what kind of angle is the fourth? Construct each of the quadrilaterals, and, of the second, construct first the convex angle.

7. How many right angles in the sum of the interior and exterior angles of a quadrilateral? How many in the exterior angles?

8. Intersect (cross) two parallel lines by two parallel lines. What relation, as to direction, do the opposite sides of the quadrilateral thus formed bear to each other? Such a quadrilateral is called a *parallelogram.*

9. Intersect two parallel lines by two others that are not parallel. How many of the sides of the quadrilateral formed are parallel? Such a quadrilateral is called a *trapezoid.*

10. If two converging or diverging lines are crossed by two other lines that are not parallel, the included quadrilateral is called a *trapezium*, of which no sides are parallel.

1. The Parallelogram.

1. Construct a parallelogram by crossing, or intersecting, two parallel lines by two other parallels.

Note. — Two angles that lie at the ends of the same side of a parallelogram are called *adjacent angles*, and the others, *opposite angles.*

2. What can you say of the sum of two adjacent angles of a parallelogram? According to what inference of parallels crossed by a transversal?

3. How do the opposite angles of a parallelogram compare in size?

4. If an angle of a parallelogram is 94°, 72°, or 64° (an oblique angle), how large, and of what kind, are the remaining angles? If one of the angles is 90°, how large, and of what kind, are the remaining angles?

Remark. — A parallelogram, all of whose angles are right angles, is a *right-angled parallelogram* or *rectangle*, and one whose angles are oblique is an *oblique-angled parallelogram* or *rhomboid.*

5. Compare the lengths of the opposite sides of a parallelogram. What general inference may be drawn from this?

6. Could you construct a parallelogram, if you had two of its sides and the included angle given? Show how you would do it.

7. What kind of figure is a quadrilateral whose opposite sides are equal?

Note. — A line connecting the opposite angles of a parallelogram is called a *diagonal.*

8. Draw the two diagonals of a parallelogram, and find into what kind of parts they divide each other. What general inference can be drawn from this division?

9. A right-angled equilateral (equal-sided) parallelogram is called a *square.* Construct such a parallelogram. Construct another, whose sides shall be equal to those of the first. Of what parts of the first did you take the measure to make the second? Of how many sides of a square is it necessary to have the length, in order to construct the square? Why?

Note. — A right-angled parallelogram whose opposite sides are equal is called a *rectangle* or *oblong.*

10. Construct a rectangle. Construct another, whose sides shall be equal to those of the first. Of how many of its sides is it necessary to have the length, in order to construct a rectangle? Why?

Note. — An oblique-angled equilateral parallelogram is called a *rhombus.*

11. Construct a rhombus. Construct another, whose sides and angles shall be equal to those of the first. What parts of the first did you measure to construct the second?

12. What must be known in order to construct a rhombus of given form and dimensions?

13. Construct a rhomboid. Construct another, whose corresponding sides and angles shall be equal to those of the first.

14. What must be known in order to construct a rhomboid of given form and dimensions?

15. In what parallelograms are the diagonals of the same length? In what, of unequal length?

16. In what parallelograms are the adjacent angles formed by the two diagonals equal, and the diagonals perpendicular to each other?

2. The Symmetrical Trapezoid.

Note. — If two parallels be crossed by two converging lines, so that the angles made with the same parallel are equal, the resulting figure is a *symmetrical trapezoid.*

1. Construct a symmetrical trapezoid, and compare the lengths (1) of the parallel sides and (2) of the converging sides. What inference do you draw concerning the lengths of the opposite sides of such a trapezoid?

2. What kind of angles are those on the longer parallel? Those on the shorter?

3. What relation do the angles on either of the oblique lines bear to each other? What is their sum? According to what inference?

4. How do the angles at each of the parallels compare with each other in size? If one of the angles is 47°, 52°, 80°, or 28°, how large is each of the others?

5. Draw the diagonals of the trapezoid, and compare their lengths. What do you find?

6. Compare the parts of the diagonals that meet the longer parallel; also those that meet the shorter. Are all of

them of the same length? If not, what difference do you find?

7. Draw perpendiculars from the ends of the shorter parallel to the longer, and compare the parts cut off from the longer. How do their lengths compare?

8. Lay off the length of the shorter of the two parallels of a symmetrical trapezoid upon the longer, to find their difference. From the point thus found draw a line which, with the difference and its adjoining oblique line, shall form a triangle. What kind of triangle, with regard to its sides, is it? What relation does the sum of its sides bear to the base? According to what inference? What relation does either of the oblique sides bear to half the base? What kind of quadrilateral is that adjacent to the triangle?

9. *Given three lines, two of them to be the parallels, and the third one of the oblique sides, to construct a symmetrical trapezoid.* — Measure off on the longer of the parallels the length of the shorter. Upon the remainder, or difference, as the base, and the third line as one of the sides, erect an isosceles triangle. Adjacent to the triangle, construct a parallelogram, two of whose sides shall be respectively the part measured off on the longer parallel and the adjoining oblique side of the triangle. The parallelogram and the triangle together form the trapezoid. Explain why. How many sides of a symmetrical trapezoid must be given to construct the trapezoid?

10. *Given an acute angle and two lines, one of the lines to serve as the measure of the shorter parallel, the other as that of one of the oblique sides, of a symmetrical trapezoid; to construct the trapezoid.* — Make one of the sides of the angle equal to the oblique line, and, with it as a radius and its end as a center, describe an arc that shall intersect the other side. Connect the intersection with the end used as center of the

arc. What kind of triangle have you formed? Why? To the base of the triangle add the parallel, and upon the part thus added erect a parallelogram in the same manner as in the previous case. What do the two figures together form? Why? How many elements (conditions or parts) of a symmetrical trapezoid are necessary to construct the trapezoid?

11. If instead of an acute angle an obtuse angle were given, how would you find one of the acute angles?

12. If you had given the two parallel sides and one of the acute angles, as elements of a symmetrical trapezoid, how would you, with the aid of the isosceles triangle, construct the trapezoid?

VI. THE POLYGON.

1. A figure that has more than four corners or angles is called a *polygon* (many corners). Triangles are sometimes called polygons.

2. Construct a five-sided figure (pentagon), and from a point within draw lines to all the corners. Into how many parts is the polygon divided? What kind of figures are the parts? How does the number of sides of the polygon compare with the number of parts? How does the number of triangles compare with the number of corners of the polygon? Would the same hold good in any polygon? Try whether it would.

3. How many right angles does the sum of all the angles of the five triangles of the foregoing polygon equal? How does the number of right angles compare with the number of corners of the pentagon? What angles of the triangles do not constitute part of the corners of the pentagon? What is their sum? What is the sum of the other angles of the triangles, those that constitute the angles of the pentagon?

How does this sum compare with the sum of all the angles of the triangles?

4. If you had given the number of corners of a polygon, could you find the number of right angles in its interior angles? How?

5. Suppose a figure has 8, 12, 15, or 18 sides, how many right angles are its interior angles equal to?

6. How many right angles in the interior and exterior angles of a pentagon? What is the sum of the exterior angles? How does this sum compare with the number of corners of the figure?

7. What is the sum of the exterior angles of a polygon having 6, 8, 10, or 16 corners?

8. With any suitable radius describe a circle, and find how many times the radius can be applied to the circumference. Connect the points with straight lines. What kind of polygon have you formed? How many more sides has it than angles? When all the sides and all the angles of a polygon are equal, the figure is called a *regular polygon.*

9. Construct a regular twelve-cornered figure (dodecagon) by dividing each of the arcs of a six-cornered figure (hexagon) into halves, and connecting the adjacent points with straight lines.

10. A straight line that passes through the center of a circle and intersects the circumference in two points is a *diameter* of the circle.

11. If you should draw two diameters of a circle at right angles to each other, and connect the adjacent ends by straight lines, what kind of figure would you form? Draw such a figure. Divide its arcs into halves, and construct the resulting figure; again divide the latter into halves, and construct the corresponding figure.

12. How many degrees in the interior angle of a regular 5, 6, 8, 10, or 15 sided polygon?

13. From the center of a regular polygon draw lines to the corners and compare their lengths. In what respects are the resulting triangles equal? What do you conclude from this? According to what inference? How do the angles at the center compare with each other? How many degrees in an angle at the center of a regular 6, 8, 9, 15, or 18 cornered figure?

VII. THE AREAS OF FIGURES INCLOSED BY STRAIGHT LINES.

1. The Rectangle, or Right-angled Parallelogram.

1. Construct a rectangle whose sides shall be 6 inches and 4 inches in length, or, if drawn upon the blackboard, larger, —say 9 inches and 5 inches respectively. Since in a parallelogram any side may be regarded as the base, the longer side (6 inches) may be taken for it; and since in a parallelogram, also, a perpendicular from one side upon the other, or upon its prolongation, may be taken as the height, the shorter side (4 inches) may in this case be taken for it. Divide the sides into inches, and connect the opposite points by straight lines.

2. Into what kind of parallelograms is the rectangle divided? How long is each side of one of the parallelograms?

3. A figure of four equal sides and with all its angles right angles is called a *square*. When the sides of a square are *an inch* in length, the figure is called a *square inch;* when a *foot* in length, a *square foot;* when a *yard*, a *square yard*, etc.

4. In the foregoing rectangle (6 by 4), (*a*) How many

of the squares lie in the first row, or the row on the base? (*b*) How many, in the second row? (*c*) How many, in the third? (*d*) How does the number in each row compare with the number of inches in the base? (*e*) How does the number of rows compare with the number of inches in width or height? (*f*) If we should multiply the number in the first row by the number of rows, how would the product differ from the number of squares in the whole figure? (*g*) If we should multiply the number of inches in the length of the figure (the base) by the number in its width or height, how would the product differ from the last found product, or area, of the figure? (*h*) How, therefore, may the area of a parallelogram or rectangle be found without dividing the figure into squares?

NOTE. — The numbers in most of the problems and exercises that follow are based upon the metric system; but the pupils may, if they prefer it, regard them as inches, feet, or other dimensions, — linear, square, etc., as the case may be.

The teacher or pupil who prefers to change the given numbers to those of the common system can easily do so by bearing in mind the following relations:

The meter	= 39.37 inches.
The decimeter ($\frac{1}{10}$ of meter)	= 3.937 inches.
The centimeter ($\frac{1}{100}$ of meter)	= .3937 inches.
The square meter	= 1550 square inches.
The cubic meter	= 35.316 cubic feet.

5. Find the areas of the following right-angled parallelograms, the base and the height of each being given.

NOTE. — m. stands for meter, cm. for centimeter, dm. for decimeter, sq. cm. for square centimeter, and cu. dm. for cubic decimeter.

	Base.	Height.		Base.	Height.
(1)	13 cm.	8 cm.	(4)	17 cm.	9 cm.
(2)	24 cm.	6 cm.	(5)	12 cm.	11 cm.
(3)	15 cm.	9 cm.	(6)	14 cm.	17 cm.

6. Find the bases of the following rectangles, the area and the height being given:

	Area.	Height.		Area.	Height.
(1)	94 sq. cm.	7 cm.	(4)	127 sq. cm.	13 cm.
(2)	108 sq. cm.	9 cm.	(5)	87 sq. cm.	9 cm.
(3)	76 sq. cm.	8 cm.	(6)	116 sq. cm.	11 cm.

7. Find the heights of the following rectangles, the area and the base being given:

	Area.	Base.		Area.	Base.
(1)	124 sq. cm.	31 cm.	(4)	91 sq. cm.	13 cm.
(2)	144 sq. cm.	24 cm.	(5)	108 sq. cm.	18 cm.
(3)	96 sq. cm.	12 cm.	(6)	119 sq. cm.	17 cm.

8. Since both dimensions of a square are the same, how would you find its area or surface?

9. The side of a square is 9, 6, 11, or 15; what is its area?

10. When the surface of a square is 144, 81, 169, 225, or 361, what is its side?

2. The Square.

1. Construct a square and prolong its base ab to c, making bc less than ab.

2. Upon ac construct a second square which shall include the square upon ab.

3. Prolong the sides of the square ab to meet those of ac. The square upon ab is written $\overline{ab}^2$; that upon ac, $\overline{ac}^2$.

4. We must now find the length of ac or $ab + bc$.

5. When $\overline{ab}^2$ is 100, 400, 900, 1600, 2500, 3600, 4900, 6400, or 8100, how long is ab?

6. After subtracting $\overline{ab}^2$ from $\overline{ac}^2$, what figures remain?

Note. — The pupils should be led to see that both of the rectangles that remain are together equal to a parallelogram whose base is twice ab, and its height bc.

7. With what numbers must the remainder be divided to obtain bc, when ab is 10, 20, 30, 40, 50, 60, 70, 80, or 90?

8. How many square meters, inches, or whatever the denomination may be, remain for the small square in the figure when *bc* is 1, 2, 3, 4, 5, 6, 7, 8, or 9?

9. What is the length of the side of a square whose surface is 4225, 7569, 1764, 6241, 9025, 3364, or 5476?

10. In the foregoing figure, prolong *ac* to *d*, making *cd* yet smaller than *bc*.

11. Upon *ad* erect a third square ($\overline{ad}^2$), which shall include $\overline{ac}^2$.

12. Prolong the sides of $\overline{ac}^2$ to meet those of $\overline{ad}^2$.

13. The problem now is to find the length of *ad* or $ab + bc + cd$.

14. How long is *ab*, when $\overline{ab}^2$ is 10000, 40000, 90000, 160000, 250000, 360000, 490000, 640000, or 810000?

15. Next find the length of *bc*. How can you do it?

16. How many square meters or inches (as the case may be) are in the middle square ($\overline{ac}^2$), where *bc* is 10, 20, 30, 40, 50, 60, 70, 80, or 90?

17. If we regard the sides of $\overline{ac}^2$ as the base, and *cd* as the height or width of the adjacent rectangles, how can we find *cd*?

18. How many square meters remain for the smallest square in the figure if *cd* is 1, 2, 3, 4, 5, 6, 7, 8, or 9?

19. What is the side of a square whose area is 24336, 34969, 60516, 47524, 141376, or 105625?

3. The Oblique-angled Parallelogram.

1. Construct a rectangle, and from one end of its base draw an oblique line to the side opposite the base. From the other end of the base draw an oblique line parallel to

the first, to meet the prolongation of the side opposite the base.

2. What kind of figure is the second one drawn?

3. In what respects do the two figures agree? *Both have the same base and lie between the same parallels.*

4. In what respects do the two triangles — the one within the rectangle, the other without — agree? What may we conclude from this? According to what inference? *Congruence.*

5. How does the area of the interior triangle of the rectangle compare with that of the exterior one? How, therefore, does the area of the oblique-angled parallelogram compare with that of the rectangle?

6. What factors must be multiplied together to obtain the area of either parallelogram, or of any parallelogram?

4. The Triangle.

1. Construct a parallelogram and, with a diagonal, divide it into two triangles.

2. In what parts or respects do the two triangles agree?

3. What conclusion do you draw from this? According to what inference?

4. What part of the parallelogram is included in each triangle?

5. In how many ways may the area of a triangle be found?

6. Find the area of each of the following triangles, the base and the altitude (height) being given:

	Base.	Altitude.		Base.	Altitude.
(1)	8 cm.	9 cm.	(4)	9 cm.	14 cm.
(2)	12 cm.	8 cm.	(5)	11 cm.	13 cm.
(3)	15 cm.	7 cm.	(6)	17 cm.	5 cm.

7. Find the bases of the following triangles, the area and altitude being given:

	Area.	Altitude.		Area.	Altitude.
(1)	144 sq. cm.	16 cm.	(4)	72 sq. cm.	9 cm.
(2)	96 sq. cm.	8 cm.	(5)	124 sq. cm.	4 cm.
(3)	124 sq. cm.	5 cm.	(6)	84 sq. cm.	12 cm.

8. Find the altitudes of the following triangles, the area and base being given:

	Area.	Base.		Area.	Base.
(1)	78 sq. cm.	16 cm.	(4)	98 sq. cm.	7 cm.
(2)	156 sq. cm.	20 cm.	(5)	128 sq. cm.	8 cm.
(3)	112 sq. cm.	14 cm.	(6)	81 sq. cm.	9 cm.

a. — Triangles of the Same Base and Equal Altitude.

1. Construct a triangle, and through its vertex draw a line parallel to the base.

2. From various points in the parallel draw lines to both ends of the base, thus also forming triangles.

3. What can you say of the areas of all these triangles? How are they found?

REMARK. — The parallel line is called the *geometrical locus* of the vertices of the triangles.

b. — The Right-angled Triangle.

1. Construct a right-angled triangle whose sides shall be to each other in the ratio of 3 to 4, — say 3 centimeters and 4 centimeters, or 3 inches and 4 inches, — and find the length of the hypotenuse.

2. Upon the hypotenuse and the sides erect squares.

3. Find the area of each of the squares.

4. How does the sum of the areas of the squares on the two sides compare with the area of the square on the hypotenuse? What general inference may be drawn from this?

5. If one side of a right-angled triangle is 5 inches and the other 12, how long is the hypotenuse?

6. What is the sum of the squares of the sides?

7. What is the square root of the sum? *The square root of the sum is the length of the hypotenuse.*

8. If the hypotenuse of a right-angled triangle is 29 inches, and one of the sides 21 inches, how long is the other side?

9. What remains after subtracting the square of the given side from that of the hypotenuse? *The square root of the remainder is the other side.*

c. — The Isosceles Triangle.

1. If in an isosceles triangle a line be drawn from the vertex to the middle of the base, thus dividing the triangle into two right-angled triangles, which line of each is the hypotenuse?

2. Which lines are the sides?

3. Find one of the sides in each of the following isosceles triangles, the base and perpendicular upon it from the vertex being given:

	Base.	Perpendicular.		Base.	Perpendicular.
(1)	12 cm.	9 cm.	(4)	13 cm.	5 cm.
(2)	17 cm.	8 cm.	(5)	24 cm.	9 cm.
(3)	15 cm.	11 cm.	(6)	18 cm.	16 cm.

4. Find the base of each of the following isosceles triangles, the perpendicular and one of the sides of each triangle being given:

	Side.	Perpendicular.		Side.	Perpendicular.
(1)	24 cm.	12 cm.	(4)	16 cm.	7 cm.
(2)	18 cm.	11 cm.	(5)	19 cm.	8 cm.
(3)	25 cm.	9 cm.	(6)	21 cm.	13 cm.

5. Find the perpendiculars of the following isosceles triangles, the base and one of the sides being given:

	Base.	Side.		Base.	Side.
(1)	24 cm.	19 cm.	(4)	16 cm.	24 cm.
(2)	17 cm.	21 cm.	(5)	18 cm.	17 cm.
(3)	22 cm.	18 cm.	(6)	12 cm.	16 cm.

6. *To construct a square equal to two given squares.* — Construct a right-angled triangle, whose sides shall be equal to the sides of the given squares. Upon the hypotenuse as a base, erect a square, and this will be equal to the sum of the given squares. Why?

7. *To construct a square equal to the difference of two given squares.* — Construct a right angle, making one of its sides equal to the side of the smaller square. With the end of this side as a center and the side of the larger square as a radius, describe an arc which shall intersect and limit the other side. A square erected on the last-named side will be equal to the difference of the given squares. Why?

5. The Trapezoid.

1. Draw a horizontal line 9 inches long; 4 inches from it draw another 7 inches long and parallel to the first. With these two lines construct a trapezoid, and with a diagonal divide it into two triangles.

2. What is the area of the triangle that has the longer parallel as a base? What, of that which has the shorter as a base?

3. What is the area of both triangles, or of the trapezoid?

4. If you should multiply the sum of both parallels by half the distance between them, what result would you obtain? Would it differ from the area of the trapezoid? Why?

5. Suppose you should multiply the half sum of the parallels by the whole distance between them, what result would you obtain? Why?

6. If you should multiply the sum of the parallels by the distance between them, and divide the product by two, what result would you obtain? Why?

7. Find the area of the following trapezoids, the length of the parallels and the distance between them being given:

	Parallels.	Dist. Betw.		Parallels.	Dist. Bet.
(1)	15 and 17 cm.	8 cm.	(4)	18 and 12 cm.	12 cm.
(2)	23 and 19 cm.	11 cm.	(5)	27 and 13 cm.	14 cm.
(3)	34 and 16 cm.	9 cm.	(6)	48 and 34 cm.	16 cm.

8. Find the distance between the parallels of the following trapezoids, the area and the lengths of the parallels being given:

	Area.	Parallels.		Area.	Parallels.
(1)	96 sq. cm.	15 and 19 cm.	(4)	186 sq. cm.	19 and 11 cm.
(2)	144 sq. cm.	28 and 20 cm.	(5)	360 sq. cm.	48 and 24 cm.
(3)	108 sq. cm.	13 and 14 cm.	(6)	216 sq. cm.	12 and 15 cm.

9. Find the length of one of the parallels of each of the following trapezoids, the area, the other parallel, and the distance between the parallels being given:

	Area.	Parallel.	Dist. Betw.		Area.	Parallel.	Dist. Betw.
(1)	129 sq. cm.	12 cm.	18 cm.	(4)	88 sq. cm.	8 cm.	10 cm.
(2)	288 sq. cm.	16 cm.	22 cm.	(5)	156 sq. cm.	12 cm.	14 cm.
(3)	168 sq. cm.	8 cm.	31 cm.	(6)	248 sq. cm.	8 cm.	36 cm.

10. *The relation of the number of sides of a figure* (1) *to the number of diagonals,* (2) *to the number of triangles into which the figure may be divided.* — Construct a seven-sided figure, and from one of its angles draw all the possible diagonals.

11. How many more sides are there than diagonals?

12. How many more sides are there than you formed triangles?

13. What inference can you draw from your figure concerning the relation of the number of sides of a figure to that of its diagonals?

14. What inference can you draw concerning the relation of the number of sides of a figure to that of the number of triangles into which the figure may be divided?

15. How many diagonals may be drawn from an angle, and how many triangles thus formed, in a figure of 4, 6, 8, 10, 12, or 14 sides? In a triangle?

6. The Polygon.

1. To find the area of a trapezium and of a polygon, divide the figure into triangles, and the sum of their areas or surfaces will be the area of the figure.

2. Find the area of each of the following regular figures, the number of sides, their length, and the perpendicular upon them from the center being given:

	No. of Sides.	Length.	Perp.		No. of Sides.	Length.	Perp.
(1)	5	3.32 m.	2.28 m.	(4)	10	2.50 m.	3.84 m.
(2)	6	75 cm.	65 cm.	(5)	12	50 cm.	93.3 cm.
(3)	8	90 cm.	1.08 cm.	(6)	16	36 cm.	32 cm.

VIII. THE CIRCLE.

1. With any assumed opening of the compasses describe a circle.

2. From the middle point, or center, of the circle draw straight lines, or radii, to the circumference and compare their lengths.

3. What can you say of the lengths of the radii?

4. Are any points of the circumference farther from the center than others? What, therefore, is the general inference?

5. How does the length of a diameter (through measuring) compare with that of a radius?

6. How do the different diameters compare with each other in length? What general inference may therefore be drawn?

7. At the end of a radius erect a perpendicular, and notice in how many points it touches the circumference. Such a perpendicular is called a *tangent* (*touching* line), and the point at which it touches the circumference is the *tangent point.*

8. At the end of a radius erect a line which shall make an oblique angle with the radius. In how many points can such a line meet, or intersect, the circumference? The name of this line is *secant* (*cutting* line), and the part within the circumference is a *chord.*

9. From the center of a circle draw a line to the middle of a chord, and compare the adjacent angles which it makes with the chord. How do the two lines meet each other? Draw an inference from this.

10. If from the center of a circle a perpendicular be drawn to a chord, into what kind of parts does it divide the chord? What general inference may be drawn from this?

11. If a perpendicular be erected at the middle of a chord, through what point in the circle will it pass? What inference may be drawn from this?

12. If two chords be drawn in a circle, and a perpendicular be erected at the middle of each, at what point in the circle will the perpendiculars meet? How, consequently, may the center of a circle be found?

13. *To draw a circumference through three points not in the same line.* — Connect the points by straight lines, and at the middle of each erect a perpendicular. The intersection

of the perpendiculars will be the center of the circle, and the lines will be chords of it. How many points, therefore, determine the circumference of a circle?

14. If you should describe two circles that have two points in common, in how many points would they intersect each other?

15. In a straight line, select three points at suitable distances apart. With the first as a center, and the distance to the second as a radius, describe a circle. Likewise, with the third as a center, and the distance to the second as a radius, describe a circle. In how many points do the circles touch? Is the tangent point within or without the circles?

16. In a straight line, select three points, and, with the first as a center and the distance to the third as a radius, describe a circle; likewise, with the second as a center, and the distance to the third as a radius, describe a circle. In how many points do the circles touch each other? Is the tangent point an interior or an exterior one?

17. The line in which the centers and the tangent point lie is called the *central line*.

1. Peripheral and Interior and Exterior Eccentric Angles.

1. Describe a circle, and upon an arc of it as a base construct an angle whose vertex shall lie in the circumference. Such an angle is called a *peripheral angle.*

2. With the radius of the foregoing circle and the vertex of the angle as a center, describe an arc that shall intersect both sides of the angle. Measure the intercepted arc, and compare its length with that upon which the angle stands. How do they compare? What inference may therefore be drawn concerning the measure of a peripheral angle when compared with a center angle?

NOTE. — A *center angle* is one whose vertex is at the center, or whose measuring arc is described with the same radius as that of the circle in which it is formed.

3. How many degrees in the arc intercepted by the sides of a peripheral angle of 24°, 38°, 49°, 57°, 68°, or 89°?

4. How many degrees in a peripheral angle that stands upon an arc of 72°, 96°, 120°, 144°, 168°, or 180°?

5. *Upon a straight line, to construct a right-angled triangle.* — Divide the given line into two equal parts. With the middle point as a center, and half of the line as a radius, describe a semicircle; chords drawn from any point in the arc to the ends of the given line will form the sides of a right-angled triangle. Why is the triangle thus formed right angled?

6. *At any point in a circumference, to draw a tangent to the circumference.* — Describe a circumference; draw a radius to the tangent point; then a perpendicular erected at its extremity, or end, will be the required tangent.

7. *From a point without a circumference, to draw a tangent to the circumference.* — From the given point draw a straight line to the center of the circle. Upon the line, as a diameter, describe a second circle, and the points in which the circumferences intersect each other will be the points at which the tangents will touch the circumference. To prove it, draw radii to the points of intersection in the first circle, and note the angles they make with the tangents. What inference can you draw from a comparison of the lengths of the tangents?

8. *In a circle, construct an interior eccentric angle* — an angle whose vertex shall be neither in the circumference nor at the center. With the same radius as that of the circle, find the measure of the angle. Prolong the sides of the angle beyond the vertex to the circumference. With

the compasses, take the length of the arc between the extended sides, add it to the arc of the opposite or vertical angle, and compare the sum with the measure of the angle. How does the measure of the angle compare with the sum of the two arcs? What general inference may be drawn from this?

9. What is the measure of an interior eccentric angle whose arcs are 58°–17°, 69°–23°, 78°–31°, 49°–13°, 85°–29°, or 106°–47°?

10. What is one of the arcs of an interior eccentric angle when the other is 59°, and the angle 69°, 57°, 98°, 117°, 123°, or 79°?

11. *Upon the arc of a circle, erect an exterior eccentric angle* — an angle whose vertex lies without the circumference. Find the measure of this angle. Measure off on the arc upon which the angle stands an arc equal to the other arc intercepted by the sides of the angle, to find the difference between the arcs. Compare the measure of the angle with this difference. Draw the inference.

12. How many degrees in an exterior eccentric angle, if the arcs between its sides are 115°–47°, 98°–17°, 101°–59°, 83°–19°, 99°–48°, or 123°–36°?

13. If one of the arcs between the sides of an exterior eccentric angle is 29°, what is the other if the angle is 24°, 36°, 48°, 19°, 67°, or 78°?

2. The Diameter and the Circumference of a Circle.

1. In the year 212 B.C., Archimedes, the greatest of ancient mathematicians, found that the circumference of a circle is very nearly 3.14 (more nearly 3.1416) times the diameter. When, therefore, the diameter is given, how is the circumference found? If the circumference is given, how is the diameter found?

2. Find the circumference of a circle whose diameter is 6, 7, 8, 12, 20, or 32 inches.

3. What is the diameter of a circle whose circumference is 50.24, 70.48, 120.36, 212.75, or 342.62 inches?

3. The Area of the Circle.

1. By means of radii, divide a circle into triangles.

2. If the arcs upon which the radii stand be considered as the bases of the triangles, what line of the circle is the sum of the bases?

3. What line is the common height or altitude of the triangles?

4. The sum of the areas of the triangles, and therefore the area of the circle, is found by multiplying the circumference by half the radius. The following is the formula:

$$C \times \frac{R}{2}.$$

Remark.—The pupils should be led to discover the method of finding the area of the circle from the sum of the triangles.

5. Since the circumference is 3.14 times the diameter, the area of the circle may also be found by taking 3.14 times the diameter and multiplying it by half the radius:

$$\frac{314}{100} \times \frac{D}{1} \times \frac{R}{2}.$$

6. Since, however, the diameter is twice the radius, the foregoing formula may be written, $\frac{314}{100} \times \frac{2\,R}{1} \times \frac{R}{2}$, and by canceling the 2's in the numerator and the denominator, may be reduced to $\frac{314}{100} \times \frac{R}{1} \times \frac{R}{1}$, or $\frac{314}{100} \times \frac{R^2}{1}$.

7. What factor must be known that the area may be found by the last formula?

8. If the diameter of a circle is 4, 8, 9, 12, 15, 18, or 20 inches, what is the area?

9. If the area of a circle is 40.18, 84.96, 184.14, 456.24, or 378.16 square inches, what is its diameter?

4. The Annulus, or Circle-ring.

1. With the same center, describe two circles of unequal diameters. Such circles are called *concentric circles.*

2. Within a circle, describe another circle which shall not have the same center as the other. Such circles are called *eccentric circles.*

3. The surface between two concentric or two eccentric circles is called an *annulus,* or *circle-ring.*

4. If you had given the area of each of two concentric or eccentric circles, how would you find the area of the circle-ring?

5. Find the areas of the following circle-rings, the diameters of the two circles being given:

Diam. Larger.	Diam. Smaller.	Diam. Larger.	Diam. Smaller.
(1) 15 cm.	8 cm.	(4) 28 cm.	13 cm.
(2) 24 cm.	17 cm.	(5) 19 cm.	6 cm.
(3) 16 cm.	9 cm.	(6) 26 cm.	12 cm.

6. The areas of two concentric circles being given, to find the width of the circle-rings:

Larger.	Smaller.	Larger.	Smaller.
(1) 452.16 sq. m.	50.24 sq. m.	(3) 803.84 sq. m.	200.96 sq. m.
(2) 615.40 sq. m.	113.04 sq. m.	(4) 1256.00 sq. m.	314.00 sq. m.

7. Given the areas of two eccentric circles, the smaller being tangent to the larger, to find the width of the circle-rings in the central lines:

	Larger.	Smaller.		Larger.	Smaller.
(1)	379.94 sq. m.	12.56 sq. m.	(3)	706.50 sq. m.	153.86 sq. m.
(2)	530.66 sq. m.	78.50 sq. m.	(4)	907.46 sq. m.	254.34 sq. m.

5. The Sector.

1. Draw two radii in a circle. The part of the surface included between the radii and the intercepted arc is called a *sector.*

2. Draw a sector whose arc is 90°. This sector, containing one fourth of the area of the circle, is called a *quadrant.*

3. A sector whose arc is 60° is called a *sextant,* and one whose arc is 45°, an *octant.*

4. Find the area of the quadrant, sextant, and octant, when the diameter of the circle is 18, 28, 48, 72, 84, or 124 cm.

5. *To find the area of a sector when the number of degrees in its arc is not an exact divisor of the circumference.* — Find the area of a sector of one degree of arc, and multiply it by the number of degrees in the given arc.

6. Find the area of each of the following sectors, the arcs and diameters being given:

	Arc.	Diameter.		Arc.	Diameter.
(1)	57°	8 cm.	(4)	87°	48 cm.
(2)	108°	18 cm.	(5)	113°	27 cm.
(3)	95°	24 cm.	(6)	46°	39 cm.

6. The Segment.

1. Draw a chord in a circle. The part of the area of the circle that lies between the chord and its arc is called a *segment.*

2. If you had given the arc, the chord, and the diameter of the circle, could you find the area of the segment? Could

you find the area of the isosceles triangle whose base is the chord?

3. Find the area of each of the following segments, the radius of the circle, the arc, the chord, and the altitude of the triangle being given:

	Radius.	Arc.	Chord.	Alt.
(1)	7.2 cm.	63°	7.5 cm.	6.1 cm.
(2)	5.6 cm.	106°	9.1 cm.	5.3 cm.
(3)	5.9 cm.	58°	5.8 cm.	5.2 cm.
(4)	10.1 cm.	53°	9.1 cm.	9 cm.
(5)	5.3 cm.	147°	10 cm.	1.9 cm.
(6)	9.3 cm.	55°	8.7 cm.	8.3 cm.

IX. THE FUNDAMENTAL MATHEMATICAL BODIES.

Introduction.

1. The floor, ceiling, and four sides of this room inclose a certain amount of space or room.

2. Any portion of space inclosed on all sides (including floor and ceiling) is called a *geometrical body*.

3. What kind of geometrical figures are the sides of this room? What kind of parallelograms are they?

4. What direction do the ceiling and the floor take? What direction with reference to each other?

5. What direction do the sides and ends take? What, with reference to each other?

6. Compare the size of the floor with that of the ceiling. Compare also the sides and the ends with each other as to size.

7. What may be said of the size of any two parallel inclosures or sides of a room?

8. In what kind of form do the floor and the sides intersect each other? In what, the ceiling and the sides?

9. The line of intersection in which two sides meet each other is called an *edge*; it is also called the *axis of the intersecting sides.*

10. What kind of angle do any two intersecting sides of this room make with each other? How many such angles can you find in the room?

11. Since the interior angle formed by two sides is a right angle, what kind of angle is the exterior angle formed by two adjacent sides of a house? How many right angles does it contain?

12. How many corners has this room? How many surfaces or sides (including floor and ceiling) form the corners?

13. Since the angles of the parallelograms that form the corner angles are right angles, what kind of angles may the corner angles be called?

14. A line connecting the middle points of two parallel sides is a *surface axis.* How many such are possible in this room?

15. A line connecting the middle points of two opposite edges or axes is a *line axis.* How many such are possible in this room?

16. A line connecting two opposite corners is a *corner axis.* How many such are possible in this room?

17. Find the area of the six sides of this room. Their sum is the whole surface of the room.

18. Find the surface area of a room whose dimensions are: length 6.5 m., width 4 m., height 3.5 m.

THE SURFACES OF GEOMETRICAL BODIES.

1. The Regular Bodies.

The Cube. — **1.** How many sides has the cube?

2. To what class of figures do the sides of the cube belong?

3. How do the sides compare in size?

4. How could you find the surface of a cube, if only its side or edge were given?

5. If the surface of a cube were given, how would you find the length of a side?

6. What is the surface of a cube whose edge is 6, 9, 12, 16, 20, or 24 cm.?

7. If the surface of a cube is 294, 864, 1014, 486, 1536, or 384 sq. cm., what is its edge?

The Tetrahedron. — The tetrahedron is a solid inclosed or bounded by four equal equilateral triangles.

Note. — An *equilateral triangle* is one having all its sides equal.

The Octahedron. — The octahedron is a solid bounded by eight equilateral triangles.

The Icosahedron. — The icosahedron is a solid bounded by twenty equal equilateral triangles. It is a solid composed of twenty equal and similar triangular pyramids whose vertices meet in a common point.

1. What factors must be given to find the area of an equilateral triangle?

2. If you had a side and the altitude of one of the triangles given, how would you find the surface of any one of the foregoing bodies?

3. If the surface of one of the bodies and the base of one of the triangles were given, how would you find the altitude of the triangle?

4. If the surface of one of the bodies and the altitude of a triangle were given, how would you find the base of the triangle?

5. Find the surface of each one of the foregoing bodies, the base and altitude of one of its triangles being given:

	Base.	Altitude.
(1)	4 cm.	3.46 cm.
(2)	6.4 cm.	5.54 cm.
(3)	8.4 cm.	7.27 cm.

6. What is the altitude of one of the triangles of an octahedron, if the surface of the body is 280.584 sq. cm., and the base of one of its triangles 9 cm.?

7. Find the base of a triangle of an icosahedron whose surface is 2216.96 sq. cm., and the altitude of one of its triangles 13.856 cm.

The Dodecahedron. — The dodecahedron is a solid bounded by twelve equal regular *pentagons* (five-sided figures).

1. How can you find the area of a pentagon?

2. If you should inscribe a circle in a pentagon, and draw straight lines from its center to the angles or corners of the pentagon, into what kind of figures would you divide the pentagon?

3. If you had given the side of one of the pentagons and the radius of the inscribed circle, how would you find the surface of the dodecahedron?

4. If you had given the surface of a dodecahedron and the side of one of its pentagons, how would you find the radius of the inscribed circle?

5. Find the surface of a dodecahedron, the side of one of whose pentagons is 8 cm., and the radius of the inscribed circle 5.5 cm.

6. The surface of a dodecahedron is 743.4 sq. cm., the side of one of its pentagons 6 cm.; what is the radius of the inscribed circle?

7. The surface of a dodecahedron is 1671.3 sq. cm., the radius of the inscribed circle of a pentagon 6.19 cm.; what is the side of a pentagon?

The Sphere, or Ball. — Not only are polygons, whose sides and angles are equal, regarded as regular figures, but likewise squares, equilateral triangles, and all solids whose surfaces are composed of equal, regular figures. The cube, tetrahedron, octahedron, icosahedron, and dodecahedron are regarded as regular solids.

1. The sphere may also be considered a regular body or solid. It is inclosed in a regularly curved surface, every point of which is equally distant from a point within called the *center*.

2. Lines drawn from the center to the surface are *radii* of the sphere. How do these compare in length? Why?

3. A straight line passing through the center, and limited at both extremities by the surface, is a *diameter* or *sphere axis*, and its terminal points or ends are called *poles*. How does the diameter compare in length with the radius? What inference may be drawn from this?

4. If you should take a straight, sharp knife and cut a sphere through its center, into what kind of parts would you divide it?

5. What kind of figure would the cut surface make?

6. In what respects would it agree, or correspond, with the sphere?

7. *Four times the area of the cut surface (great circle) is the surface of the sphere.*

8. If the diameter alone of a sphere were given, how would you find the surface?

9. On page 43 we have a formula for finding the area of a circle. If in that formula we substitute the square of half the diameter for the square of the radius, we have for the surface of the sphere the following formula:

$$\frac{314}{100} \times \frac{D}{2} \times \frac{D}{2} \times \frac{4}{1}, \text{ or } \frac{314}{100} \times \frac{D^2}{1}.$$

10. If the surface of a sphere were given, how would you find the diameter?

11. Find the surface of a sphere whose diameter is 6, 8, 16, 18, 20, or 24 cm.

12. The diameter of a ball is 3 inches; what is its surface?

13. If the surface of a sphere is 96.5843, 132.6894, or 984.9876 sq. cm., what is its diameter?

2. The Half-regular, Uniformly-thick, Bodies.

The Pillar or Prism. — **1**. Examine a prism, and observe how many regular figures its surface has.

2. Of what kind are they in a three, four, five, or six-sided prism?

3. How do the parts of each compare in size?

4. How can we find the sum of the surfaces of all the figures of each?

5. By how many surfaces is each of the prisms bounded?

6. Of which of them are the surfaces inclined toward each other?

7. What kind of figures are the inclined surfaces? How can you tell?

8. How do these surfaces compare with each other in form and size?

9. What factor have the sides and the bases of the prisms in common?

10. What is the entire surface of a triangular prism, the side of whose base (side of triangle) is 5 cm., the perpendicular upon it from the vertex of the opposite angle 4.4 cm., and the height of the prism 18 cm.?

11. What is the entire surface of a triangular prism, when the side of the base is 12 cm., the perpendicular from the vertex of the opposite angle 10.4 cm., and the height of the prism 3.6 cm.?

12. What is the surface of a square prism or pillar, the side of whose base is 12 cm. and height 80 cm.?

13. What is the surface of a regular five-sided (pentagonal) column whose height is 80 cm., side of base 9.6 cm., and the perpendicular upon one of the sides from center 6.6 cm.?

14. What is the entire surface of a regular hexagonal (six-sided) column whose height is 56 cm., length of a side of the base 6 cm., and the radius of the inscribed circle, 5.196 cm.?

Remark. — Since prisms are inclosed by two equal regular figures and by equal rectangles, they are called *half-regular* bodies; and since their perimeters (measures around) are everywhere the same, they are called *uniformly thick* bodies.

The Round Pillar or Cylinder. — 1. How many equal faces or surfaces has the cylinder?

2. What kind of figures are they?

3. How would you find the sum of their surfaces?

4. What kind of figure would the surface between the ends form if it were unrolled? How would you find its area?

5. What factor have the ends and the side surface in common?

REMARK. — When the cylinder is lying horizontally it is called a *roller*, and when it stands vertically, a *round pillar*.

6. The diameter of a cylinder is 12 cm. and its height 80 cm.; what is its surface?

7. A roller is 120 cm. in length and 18 cm. in thickness; what is its surface?

8. What is the surface of a roller that is 12 cm. in thickness and 84 cm. in length?

9. What is the surface of a cylinder whose diameter is 18 cm. and height 1.6 cm?

3. THE HALF-REGULAR, TAPERING OR POINTED BODIES.

The Pyramid. — 1. How many regular figures has a pyramid?

2. What kind of regular figures are found in a three, four, five, or six sided pyramid?

3. How would you find their area, or surface?

4. How many separate surfaces has each of the foregoing pyramids?

5. Of which of them are the sides inclined towards each other?

6. What kind of triangles are the sides of the pyramid ?

7. How can the sum of their surfaces be found ?

8. What factor is common to the base and the sides ?

9. Find the surfaces of the following triangular pyramids, — a side of the base, the altitude or perpendicular of basal triangle, and the slant height being given :

	Side.	Altitude.	Slant Height.
(1)	8 cm.	6.93 cm.	18 cm.
(2)	39 cm.	33.775 cm.	84 cm.
(3)	7.4 cm.	6.4 cm.	24.8 cm.

10. Find the surfaces of the following square pyramids, — a side of the base and the slant height being given :

	Side.	Slant Height.
(1)	6 cm.	25 cm.
(2)	12.8 cm.	34.5 cm.
(3)	5.8 cm.	15.4 cm.

11. Find the surfaces of the following regular pentagonal (five-sided) pyramids, — a side of the base, the radius of the inscribed circle, and the slant height being given :

	Side of Base.	Radius.	Slant Height.
(1)	12.8 cm.	8.8 cm.	72 cm.
(2)	25.4 cm.	17.48 cm.	92 cm.

12. Find the surfaces of the following regular hexagonal (six-sided) pyramids, — a side of the base, the radius of the inscribed circle, and the slant height being given:

	Side of Base.	Radius.	Slant Height.
(1)	2.4 cm.	2.0784 cm.	56 cm.
(2)	7 cm.	6.06 cm.	36 cm.

Remark. — The base alone of the pyramid is a regular figure, the sides being isosceles triangles. The pyramid is also a half-regular body ; but, since its triangular sides taper to a point, it is also called a *half-regular pointed body.*

A perpendicular from the apex to the base is the *altitude*, and is called the *axis of the pyramid.*

The Cone. — **1.** How many surfaces has the cone?

2. What kind of figure is the base?

3. How is the area of the base found?

4. What kind of figure is the convex surface?

5. What kind of figure would it resemble if it were unrolled?

6. How is the area of the convex surface found?

7. To what line of the base does the curve of the convex surface correspond?

NOTE. — A line from the vertex of the cone to the middle of the base is the *axis of the cone*, and, in a perpendicular cone, is also the altitude.

The *slant height* of a cone is the distance from the apex to the circumference of the base.

8. What is the surface of each of the following cones, — the diameter of the base and the altitude (height) being given?

	Diam. of Base.	Altitude.		Diam. of Base.	Altitude.
(1)	4 cm.	9 cm.	(4)	8.6 cm.	36 cm.
(2)	6 cm.	19.4 cm.	(5)	12 cm.	52 cm.
(3)	7.5 cm.	10.6 cm.	(6)	15.5 cm.	62 cm.

9. What is the surface of a cone, the circumference of whose base is 15.7 cm. and altitude 48 cm.?

4. The Truncated, or Shortened, Bodies.

NOTE. — If from a pointed body (pyramid or cone) a piece be cut off parallel to the base, the remainder is a *shortened body*.

The only shortened bodies that will be considered are the square pyramid and the cone.

The Shortened Square Pyramid. — **1**. By how many surfaces is the shortened square pyramid bounded?

2. What kind of figures are the ends or bases?

3. In what respects do they differ?

4. What factors are necessary to determine their surfaces?

5. Of what kind of figures are the sides composed?

6. How do the sides compare in size?

7. What factor is yet necessary to determine their surfaces?

8. Find the surfaces of the following shortened square pyramids, — a side of the lower base, a side of the upper base or end, and the slant height being given:

	Side of Lower Base.	Side of Upper Base.	Slant Height.
(1)	3 cm.	2 cm.	12 cm.
(2)	5 cm.	4 cm.	16 cm.
(3)	5.8 cm.	4.2 cm.	12.5 cm.

The Shortened Cone. — **1**. By how many surfaces is the shortened cone bounded?

2. What kind of figures form the top and the bottom?

3. In what respects do the top and the bottom differ?

4. How is the surface of each found?

5. What kind of figure would the convex envelope form if it were unrolled upon a plane surface? *A trapezoid.*

6. What kind of lines would the parallel sides form? *Arcs of circles.*

7. To what lines are these arcs equal?

8. How, by means of these arcs, may the convex surface be found?

9. Find the surfaces of the following shortened cones,—the diameter of the lower base, of the upper base, and the slant height being given:

	Lower Diam.	Upper Diam.	Slant H.		Lower Diam.	Upper Diam.	Slant H.
(1)	9 cm.	5 cm.	24 cm.	(4)	3.8 cm.	3 cm.	5.6 cm.
(2)	1.4 cm.	.8 cm.	16 cm.	(5)	4.8 cm.	4.6 cm.	4.2 cm.
(3)	2.4 cm.	.3 cm.	3.8 cm.	(6)	4.4 cm.	4 cm.	1.64 cm.

THE CONTENTS OF SOLIDS, OR BODIES.

1. Bodies of Uniform Thickness.

1. What is the length of a side of a cubic inch? Of a cubic foot? A cubic centimeter? A cubic decimeter? A cubic meter?

2. How many cubic centimeters can stand by the side of one another on the base of any body or solid?

3. How high is one layer of cubic centimeters?

4. How many such layers are necessary to fill any portion of space?

5. How would you find the cubic contents of any portion of space?

6. Find the contents of a cube whose side is 8, 9, 10, 12, 14, or 16 cm.

7. Find the cubic contents of the following bodies:

(*a*) A regular triangular prism, a side of whose base is 8 cm., perpendicular of basal triangle from vertex of an angle to opposite side 6.9 cm., and height of prism 24 cm.

(*b*) A regular pentagonal column whose height is 134 cm., a side of base 8 cm., and radius of inscribed circle 5.5 cm.

(*c*) A regular hexagonal column whose height is 120 cm., a side of base 8.4 cm., and radius of inscribed circle 7.3 cm.

(*d*) A roller whose length is 6.4 cm. and diameter 13 cm.

8. The cubic contents of a roller 9 dm. in length are 113.04 cu. dm., what is its diameter?

9. What are the contents of the solid part of a hollow cylinder whose diameter is 6.4 cm., diameter of hollow part 5 cm., and height 120 cm.?

2. Tapering or Pointed Bodies.

1. The cubic contents of a pointed or tapering body are one third of those of a regular solid of the same base and altitude as the tapering body.

2. Find the cubic contents of the following bodies:

(*a*) A regular triangular pyramid, — a side of the base, perpendicular upon side from vertex of opposite angle, and altitude of pyramid being given:

	Side of Base.	Perpendicular.	Altitude.
(1)	4.6 cm.	3.98 cm.	8 cm.
(2)	92.3 cm.	79.9 cm.	360 cm.

(*b*) A square pyramid, — a side of its base and the altitude being given:

	Side of Base.	Altitude.
(1)	7 cm.	17 cm.
(2)	82 cm.	240 cm.

(*c*) A regular pentagonal pyramid whose side of base, radius of inscribed circle, and altitude are given.

	Side of Base.	Radius.	Altitude.
(1)	4.8 cm.	3.3 cm.	36 cm.
(2)	34.5 cm.	23.7 cm.	180 cm.

(*d*) A regular octagonal (eight-sided) pyramid whose altitude is 35 cm., a side of base 6.4 cm., and radius of inscribed circle 7.73 cm.

(*e*) A cone whose diameter and altitude are given.

	Diameter.	Altitude.
(1)	23 cm.	69 cm.
(2)	3.8 dm.	74 dm.

3. Cubic Contents of Truncated, or Shortened, Bodies.

1. The cubic contents of a shortened cone may be found with sufficient exactness for all practical purposes (approximately), when the difference between the upper and the lower surface is small, by finding the middle diameter between the two ends (half their sum), and treating the body as a cylinder whose diameter is the middle diameter and height that of the shortened cone.

2. If the lower diameter of a shortened cone is 4 cm., the upper 3 cm., and the height 6 cm.; what are its contents?

3. The result by the foregoing method is a little too small. When exactness is required, the following method may be pursued: From the whole cone subtract the part cut off. But to subtract the part cut off, the contents and altitude of the whole cone must be found. The altitude may be found by the following proportion: As the difference of the radii is to the larger radius (radius of cone), so is the height of the shortened part to the height of the whole cone.

The height of the whole cone may be directly found by the rule derived from the foregoing proportion, namely: Multiply the larger radius by the height of the shortened body, and divide the product by the difference between the two radii.

4. Taking the foregoing problem, we have $2 \times 6 \times \frac{2}{1} = 24$, the height of the whole cone. Subtracting from this the height of the shortened part, 6, we have 18, the height of the part cut off. To find the contents of the whole cone, we have $2 \times 2 \times 3.14 \times \frac{24}{3}$. To find the contents of the part cut off, we have $\frac{3}{2} \times \frac{3}{2} \times 3.14 \times \frac{18}{3}$, and the difference is the shortened part.

5. Find the contents of the following shortened cones by both methods:

	Larger Diam.	Shorter Diam.	Height.		Larger Diam.	Shorter Diam.	Height.
(1)	8 cm.	6 cm.	9 cm.	(4)	24 cm.	22 cm.	180 cm.
(2)	6 cm.	4 cm.	9 cm.	(5)	48 cm.	44 cm.	120 cm.
(3)	28 cm.	26 cm.	98 cm.	(6)	38 cm.	36 cm.	58 cm.

6. The contents of a barrel may also be found by the foregoing method; the larger diameter being the one at the bung (half-way between the ends), the shorter that at the ends, and the height the half length of the barrel.

7. Find the contents of the following barrels by both of the foregoing methods:

	Bung Diam.	End Diam.	Length.
(1)	38 cm.	34 cm.	46 cm.
(2)	48 cm.	45 cm.	58 cm.
(3)	38 cm.	35 cm.	42 cm.

8. The contents of a shortened square pyramid may, for all practical purposes, be found by multiplying the half sum of the areas of the ends by the height.

9. What are the contents of a shortened square pyramid whose height is 8 dm., side of base 3.8 dm., and of top 3.4 dm.?

10. When exactness is required, the following formula should be used, in which A stands for the area of the larger end, a for that of the smaller, S for a side of the larger, s for that of the smaller, and H for the height:

$$[A + a + (S \times s)]\frac{H}{3}.$$

REMARK. — In this formula, the expression $(S \times s)$ stands for the square root of the product of the areas of the ends.

11. With this formula, find the contents of the following shortened square pyramids:

	Area of Larger End.	Area of Smaller End.	Height.
(1)	12 cm.	10 cm.	45 cm.
(2)	16 cm.	14 cm.	58 cm.

12. If the end surfaces of a shortened pyramid are not squares, but three, five, or more sided figures, to obtain exactness the following formula must be used:

$$[A + a + \sqrt{(A \times a)}]\frac{H}{3}.$$

4. Regular Bodies.

1. The tetrahedron (four-surfaced body) belongs to the tapering or pointed bodies. Its contents are therefore found by multiplying its base by a third of its height.

2. Find the contents of the following tetrahedrons:

Side of Base.	Perpend. of Basal Triangle.	Altitude.
(1) 4 cm.	3.46 cm.	3.26 cm.
(2) 5 cm.	4.33 cm.	4.08 cm.
(3) 5 cm.	5.2 cm.	4.9 cm.

3. The octahedron (eight-surfaced body) can be divided into eight equal pyramids, all of whose vertices meet at the middle point of the body, and all of the pyramids having equal bases and equal altitudes. The contents of any of the pyramids are found by multiplying the base by a third of the altitude, and the contents (of the whole body) of all the pyramids, by multiplying the contents of one of them by 8, the number of them. The contents of the octahedron may also be found by multiplying the sum of the bases of the pyramids (the entire surface of the body) by a third of the height of one of them. But since the height of a pyramid is the same as a half of its surface axis, the contents of the octahedron may be found by multiplying its surface by a sixth of its surface axis. For the same reason, the contents of an icosahedron (twenty plane-faced) and of a dodecahedron (twelve plane-faced) may be found by multiplying the surface by a sixth of the surface axis.

5. The Ball, or Sphere.

4. The last of the regular bodies is the ball, or sphere. This body may be considered as composed of an infinite number of pyramids, the sum of whose bases constitutes the surface, and the height or altitude of the pyramids the radius of the ball. The contents of a ball are therefore equal to the product of its surface by a third of its radius or a sixth of its diameter.

5. Since the surface of a ball is found by the following formula: $\frac{314}{100} \times \frac{D}{1} \times \frac{D}{1}$, the cubic contents are found by the following:

$$\frac{314}{100} \times \frac{D}{1} \times \frac{D}{1} \times \frac{D}{6} = \frac{314}{600} \times \frac{D}{1} \times \frac{D}{1} \times \frac{D}{1} = \frac{314}{600} \times \frac{D^3}{1}.$$

6. Find the contents of a ball whose diameter is 6, 9, 12, 18, 30, 42, or 56 cm.

7. If the earth were a perfect sphere, and its diameter 8000 miles, what would be its cubic contents?

8. Find the cubic contents of the following shells of hollow balls whose inner and outer diameters are given:

	Outer Diameter.	Inner Diameter.
(1)	10 cm.	8 cm.
(2)	5 cm.	3.5 cm.
(3)	9 cm.	7.5 cm.

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